- AUXILIARY ELECTRONIC MATERIAL -

MATHEMATICAL APPENDIX

Getting from Eq. (1) to Eq. (2)

By definition:

$$\begin{aligned}
\Pr(p) &= \langle \delta\{p - \sigma(\mathbf{r})\}\rangle && (11)\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \langle \exp[ik\{p-\sigma(\mathbf{r})\}]\rangle && (12)\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \langle \exp[ik\{p-p_0-\Delta\sigma(\mathbf{r})]\rangle && (13)
\end{aligned}$$

Now expand the exponential in a Taylor series with respect to the small parameter $\Delta\sigma(\mathbf{r})$

$$\begin{aligned}
\Pr(p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp[ik\{p-p_0\}]\\
&\quad \times \left\langle 1 - ik\Delta\sigma - \frac{k^2\Delta\sigma(\mathbf{r})^{\mathbf{2}}}{2} + \mathcal{O}\left(k^3\Delta\sigma^3\right)\right\rangle && (14)
\end{aligned}$$

The first moment of $\Delta\sigma$ disappears by definition. The remaining terms in $\langle\dots\rangle$ can be written as $\exp[-k^2\langle\Delta\sigma^2/2\rangle]$ plus corrections of order $\Delta\sigma^3$ (or alternatively the third-order cumulant). While the derivation is exact only up to third order, there are many higher-order terms that turn out reasonably accurate not quite by coincidence, i.e., the expansion can be shown to be exact if $\Delta\sigma$ is a Gaussian random variable. Thus, $\Pr(p)$ can be approximated by the Gaussian integral

$$\Pr(p) \approx \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \exp[ik\{p-p_0\} - k^2\langle\Delta\sigma^2\rangle/2], \qquad (15)$$

whose solution is given in Eq. (2).

Elastic energy of a semi-infinite solid and Eq. (3)

The only possibility to write the elastic energy per surface area of a semi-infinite, isotropic elastic solid is

$$\frac{E}{A} = \frac{1}{2}\sum_{\mathbf{q}} q\tilde{E}|\tilde{u}(q)|^2 \qquad (16)$$

if the elastic properties of the solid are fully contained in a parameter $\tilde{E}$ with the unit of pressure (Pascal). The field conjugate to the displacement u, say $\sigma(\mathbf{r})$, will be given by $\delta E/\delta u(\mathbf{r})$ or in reciprocal space by $\partial E/\partial \tilde{u}(\mathbf{q})$, i.e.,

$$\tilde{\sigma}(\mathbf{q}) = q\tilde{E}\,\tilde{u}(\mathbf{q}). \tag{17}$$

Analyzing the unit of $\tilde{\sigma}(\mathbf{q})$ reveals that it can indeed be associated with a stress field. Comparison of this equation with Eq. (3) in this manuscript and with Eq. (12) in Ref. 8 yields $\tilde{E} = E'/2$, where E' is defined in the main text here and substitutes the term $(M_{zz})^{-1}$ used in Ref. 8. See also the dispersion relations for the generalized stiffness coefficient Φ_{zz} in Fig. 1 of Ref. 19, which also increase linearly with q at small q.

Getting to Eq. (10)

Starting point is the following representation of E

$$\begin{aligned} E \;=\; & v_0\, e^{-z_0/\zeta} \int d^2r\, e^{\{h(\mathbf{r})-u(\mathbf{r})\}/\zeta} \\ & + \frac{A}{4}\sum_{\mathbf{q}} qE'\,|\tilde{u}(\mathbf{q})|^2 + \frac{1}{A}\int d^2r\, p_0 z(\mathbf{r}), \end{aligned} \tag{18}$$

which is a combination of Eqs. (6) and (7). Keep in mind that $z(\mathbf{r}) = z_0 + u(\mathbf{r})$ with $\tilde{u}(\mathbf{q}=0) = 0$. Also, the coordinate system has been chosen to have $\tilde{h}(\mathbf{q}=0)$ disappear. Taking the derivative of E with respect to z_0 yields Eq. (8), since the elastic energy does not depend on z_0 but only on $\tilde{u}(\mathbf{q} \neq 0)$.

Now express the interaction potential in a 2nd-order cumulant expansion, see Eq. (9). Taking the derivative with respect to an arbitrary $\tilde{u}(\mathbf{q}')$ yields:

$$\begin{aligned} \frac{1}{A}\frac{\partial E}{\partial \tilde{u}(\mathbf{q})} \;=\; & \frac{\tilde{h}(\mathbf{q}) - \tilde{u}(\mathbf{q})}{\zeta^2} \times v_0\, e^{\left\{-\frac{z_0}{\zeta} + \sum_{\mathbf{q}} \frac{|\tilde{h}(\mathbf{q})-\tilde{u}(\mathbf{q})|^2}{2\zeta^2}\right\}} \\ & + \frac{qE'}{2}\tilde{u}(\mathbf{q}'). \end{aligned} \tag{19}$$

The first term on the r.h.s. of this equation can be simplified with the help of Eq. (8). Requiring $\partial E/\partial \tilde{u}(\mathbf{q})$ to disappear means:

$$0 = \left\{\tilde{h}(\mathbf{q}') - \tilde{u}(\mathbf{q}')\right\} \frac{p_0}{\zeta} + \frac{qE'}{2}\tilde{u}(\mathbf{q}'). \tag{20}$$

This equation can be solved for $\tilde{u}(\mathbf{q}')$ and the result is given in Eq. (10).

Relation to diffusion equation representation

A detailed discussion will be given in a future publication. Here it shall only be sketched that the link between the diffusion equation representation and the cumulant expansion is an integral equation for the pressure distribution, similar to the one stated by W. Manners and J. A. Greenwood [11] and by Campana *et al.* [10]. To make use of the cumulant expansion approach in a rigorous way, it will be necessary to introduce a conditional or transition probability $T(p+\eta+\Delta\eta|p-\Delta p,\eta)$, which states how likely it is to observe a stress p at the new magnification $\eta+\Delta\eta$ given that we observed a stress $p-\Delta p$ at the old magnification η.

$$\Pr(p,\eta+\Delta\eta) = \int_{-\infty}^{+\infty} d(\Delta p) T(p,\eta+\Delta\eta|p-\Delta p,\eta) \Pr(p-\Delta p,\eta). \tag{21}$$

If we knew $T(\dots)$ then we could solve $\Pr(p,\eta)$ via this integral equation.

Similar to the formal definition of the stress distribution one can define $T(\dots)$ with the help of δ functions to be:

$$T(p,\eta+\Delta\eta|p-\Delta p,\eta) \;=\; \frac{\langle\delta\{p-\sigma(\mathbf{r},\eta+\Delta\eta)\}\delta\{p-\Delta p-\sigma(\mathbf{r},\eta)\}\rangle}{\langle\delta\{p-\Delta p-\sigma(\mathbf{r},\eta)\}\rangle} \tag{22}$$

where $\sigma(\mathbf{r},\eta)$ is the stress at $\mathbf{r}$ seen at magnification η.

$T(\dots)$ can be treated in terms of a cumulant expansion just like $\Pr(p)$. For colored noise, it is possible to show that the local expectation value of σ does not change upon a change in magnification η and that the expected variance $\langle\Delta\sigma^2\rangle$ does not depend on σ. See, for example, the not quite rigorous derivation of Persson theory in Ref. [10], in which Persson's diffusion equation is obtained from the integral equation. This equation reads

$$\frac{\partial \Pr(p,\eta)}{\partial\eta} = \frac{1}{2}D(\eta)\frac{\partial^2 \Pr(p,\eta)}{\partial\sigma^2}, \tag{23}$$

where $D(\eta)=\langle\Delta\sigma^2\rangle$. Its solution to it is given in Eq. (2).